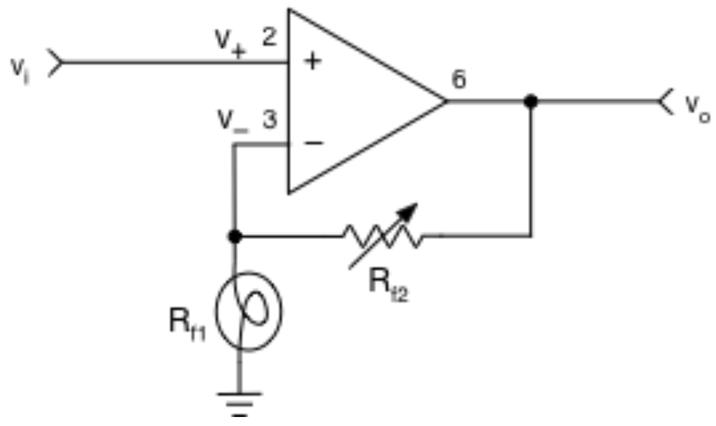
Wien Bridge Oscillator

Conventional analysis

Consider the amplifier shown below.



The potential at the noninverting input is equal to the input signal:

$$v_+ = v_i.$$

The potential at the invertiing input is related to the output via a voltage-divider:

$$v_- = \frac{R_{f1}}{R_{f1} + R_{f2}} v_o.$$

Using the ideal op-amp assumption that $v_+ = v_-$ leads to

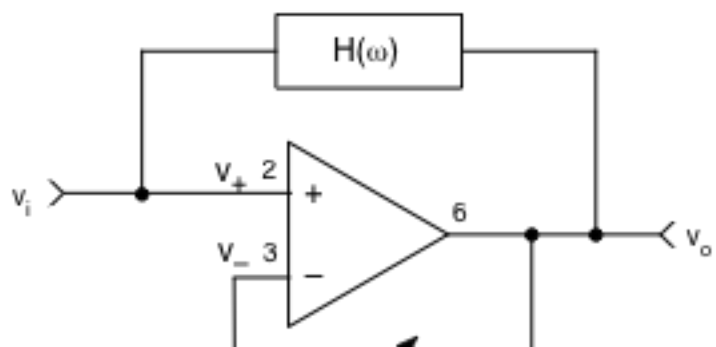
$$v_o = G v_i$$

where

$$G \equiv 1 + \frac{R_{f2}}{R_{f1}}.$$

This is the standard non-inverting amplifier configuration.

Now let's ask if we can sustain a finite output if, instead of an external input, we feed the output back to the input through a frequency-dependent network.



Using complex phasor notation such that $v = \mathbb{R}\hat{V}e^{j\omega t}$ (where $\mathbb{R}$ means "real part of") , we write

$$\hat{V_i} = H(\omega)\hat{V_o}.$$

We also had

$$\hat{V_o} = G(\omega)\hat{V_i}.$$

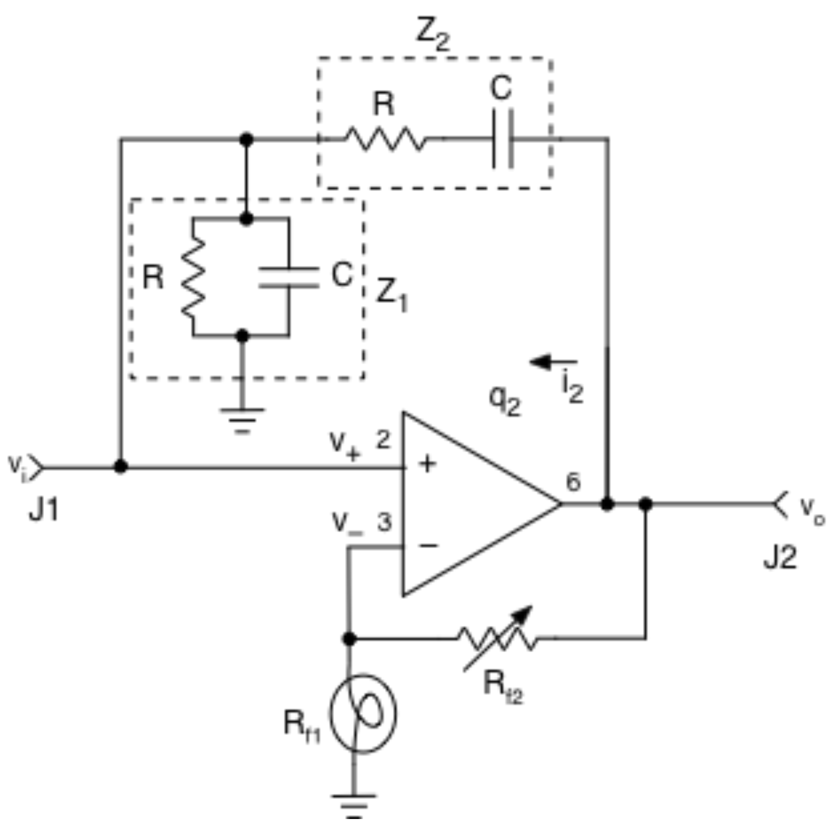
(Note: in the current case G actually does not have any frequency dependence. The notation is for purposes of generality.) Self-consistency requires

$$\hat{V_i} = G(\omega)H(\omega)\hat{V_i}.$$

This requires that the loop gain

$$G(\omega)H(\omega) = 1.$$

This is the Barkhausen condition for oscillation, which implies both that the magnitude of the loop gain is unity and that the phase shift is zero or a multiple of 2π .



Wien Bridge

Consider the resistance - capacitance network shown above.

$$\hat{V_i} = \frac{Z_1}{Z_1 + Z_2}\hat{V_o}.$$

$$\hat{V_i} = \frac{\frac{R}{1+j\omega RC}}{\frac{R}{1+j\omega RC} + (R - j\frac{1}{\omega C})}\hat{V_o}.$$

$$\hat{V_i} = \frac{1}{3 + j\left(\omega RC - \frac{1}{\omega RC}\right)}\hat{V_o}.$$

Thus

$$H(\omega) = \frac{1}{3 + j\left(\omega RC - \frac{1}{\omega RC}\right)}.$$

Remember that

$$\hat{V_o} = G\hat{V_i}$$

with G having the real value

$$G = 1 + \frac{R_{f2}}{R_{f1}}$$

.

So the loop gain requirement

$$GH(\omega) = 1$$

becomes

$$\frac{G}{3 + j\left(\omega RC - \frac{1}{\omega RC}\right)} = 1.$$

This is true provided

$$G = 3.$$

$$\omega = \frac{1}{RC}.$$

These then are the conditions for oscillation with this network which is called the Wien bridge.

Note that the above conditions then require

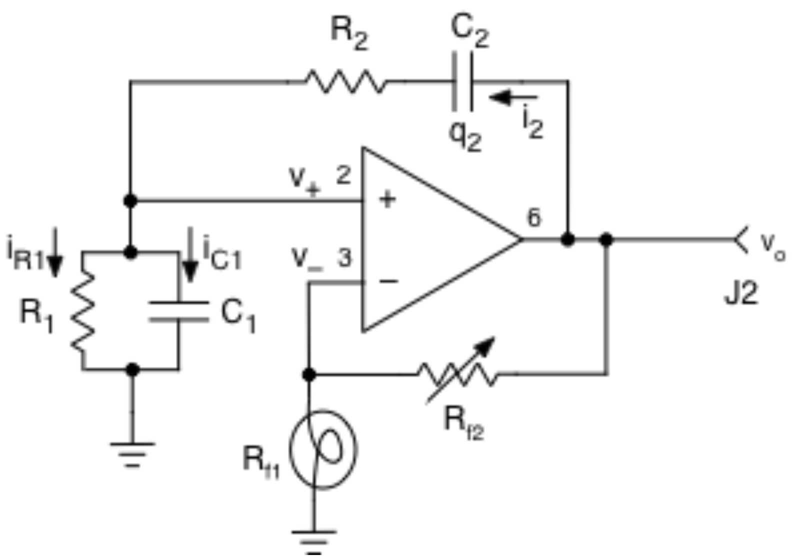
$$\frac{R_{f2}}{R_{f1}} = 2.$$

In the practical circuit, we will select resistor R_{f2} so that it can be varied until sufficient gain is reached to start oscillations. We consider R_{f2} to be the *bifurcation parameter* of the system.

Time-Domain Analysis

The above criteria are useful in creating the oscillator design. However, insight into the oscillator dynamics - leading to an understanding of what controls the amplitude and shape of oscillation - requires a time-domain analysis.

Consider the same circuit, re-drawn below with currents and voltages indicated at various points in the circuit.



Linear circuit analysis

We assume that all components have constant values. Later, the analysis will be generalized to allow the feedback resistor R_{f1} to increase with potential drop across the resistor due to self-heating, introducing nonlinearity into the system and thereby limiting the amplitude of oscillation.

Signal at the non-inverting input

Let's first first sum potential drops across the R_2C_2 branch.

$$v_+ + i_2R_2 + \frac{1}{C_2}q_2 = v_o.$$

Differentiate this equation and use the fact that $i_2 = dq_2/dt$:

$$\frac{dv_+}{dt} + R_2\frac{di_2}{dt} + \frac{1}{C_2}i_2 = \frac{dv_o}{dt}.$$

Now from Kirchoff's Law for the sum of currents into a junction, using the ideal op amp behavior that zero current flows into the non-inverting (+) input:

$$i_2 = i_{R1} + i_{C1}.$$

Thus

$$i_2 = \frac{1}{R_1}v_+ + C_1\frac{dv_+}{dt}.$$

Inserting this into equ. [xx] we get:

$$\frac{dv_+}{dt} + R_2\frac{d}{dt}\left(\frac{1}{R_1}v_+ + C_1\frac{dv_+}{dt}\right) + \frac{1}{C_2}\left(\frac{1}{R_1}v_+ + C_1\frac{dv_+}{dt}\right) = \frac{dv_o}{dt}.$$

Collecting terms:

$$\left[R_2C_1\frac{d^2}{dt^2} + \left(1 + \frac{R_2}{R_1} + \frac{C_1}{C_2}\right)\frac{d}{dt} + \frac{1}{R_1C_2}\right]v_+ = \frac{dv_o}{dt}.$$

Signal at the inverting input

At the inverting input we have

$$v_- = \frac{R_{f1}}{R_{f1} + R_{f2}}v_0.$$

$$v_- = \frac{1}{1 + \frac{R_{f2}}{R_{f1}}}v_0.$$

$$v_- = \frac{1}{G}v_0,$$

where the gain is given by

$$G \equiv 1 + \frac{R_{f2}}{R_{f1}}.$$

Combined results

Now use again use ideal op-amp behavior to require that

$$v_+ = v_-,$$

so that in the above differential equation we can substitute for v_+ using

$$v_+ = v_- = \frac{1}{G}v_0.$$

Thus we re-write the original differential equation entirely in terms of v_0 :

$$\left[R_2C_1\frac{d^2}{dt^2} + \left(1 + \frac{R_2}{R_1} + \frac{C_1}{C_2}\right)\frac{d}{dt} + \frac{1}{R_1C_2}\right]\frac{1}{G}v_o = \frac{dv_o}{dt}.$$

Multiply through by G and re-arrange terms:

$$\frac{d^2v_o}{dt^2} + \left(1 + \frac{R_2}{R_1} + \frac{C_1}{C_2} - G\right)\frac{1}{R_2C_1}\frac{dv_o}{dt} + \frac{1}{(R_1C_2)(R_2C_1)}v_o = 0.$$

If we match resistors so that $R_1 = R_2 \equiv R$ and capacitors so that $C_1 = C_2 \equiv C$ then this simplifies to:

$$\frac{d^2 v_o}{dt^2} + (3 - G)\frac{1}{RC}\frac{dv_o}{dt} + \frac{1}{(RC)^2} v_o = 0.$$

Solving the linear equation

Again, assuming that all of the coefficients are constant, we can insert a soluton $v_o = V_o e^{st}$ leads to the characteristic equation,

$$s^2 + (3 - G)\frac{1}{RC} s + \frac{1}{(RC)^2} = 0.$$

The solution is

$$s = \frac{(G - 3) \pm \sqrt{(G - 3)^2 - 4}}{2} \omega_0.$$

where $\omega_0 \equiv 1/RC$. The values are complex for $1 < G < 5$ and the solution grows exponentially when $G > 3$. Of course, indefinite exponential growth is not possible; eventually the power supply limits are reached. However, if an introduction of nonlinearity causes the gain to diminish with growing amplitude the oscillations will saturate. In such a case we have a bifurcation to an oscillatory solution at $G = 3$. This occurs when

$$\frac{R_{f2}}{R_{f1}} = 2,$$

for which case

$$s = \pm j\omega_0.$$

Note that we can re-write the differential equation (flipping the middle term to conform better with the form of the Van der Pol equation - see below) as:

$$\frac{d^2 v_o}{dt^2} - \left(\frac{R_{f2}}{R_{f1}} - 2\right)\omega_0 \frac{dv_o}{dt} + \omega_0^2 v_o = 0.$$

In [2]:

```
# TO BE ADDED Code to calculate eigenvalues as a function of feedback resistor ratio
```

Amplitude-dependent gain

We will now model the feedback resistor R_{f1} as a lamp filament that heats up as the output v_0 increases from zero. As the filament temperature T increases from a reference temperature T_0,

$$R_{f1}(T) = R_{f10}[1 + \alpha(T - T_0)].$$

We need to model how the temperature of the lamp filament varies with power P dissipated within the filament, which in turn depends on the potential drop across the filament and temperature-dependent resistance of the filament.

$$P = (i_{f1})^2 R_{f1}(T) = \left(\frac{v_o}{R_{f1}(T) + R_{f2}}\right)^2 R_{f1}(T) = \frac{R_{f1}(T)}{\left[R_{f1}(T) + R_{f2}\right]^2} v_o^2.$$

Suppose the filament has heat capacity C_{fil}. Let us assume that the heat transport away from the filament is proportional to the difference between the filament temperature T and the ambient temperature T_a:

$$\dot{Q}_{\text{transfer}} = \frac{1}{\mathfrak{R}}(T - T_a),$$

where $\mathfrak{R}$ is the effective thermal resistance of the bulb. Even radiative transport is modeled this way near onset when the temperature difference is still modest relative to the initial absolute temperature (about 300K):

$$\varepsilon\sigma(T^4 - T_a^4) = \varepsilon\sigma[T_a + (T - T_a)]^4 - T_a^4 \approx 4\varepsilon\sigma T_a^3 (T - T_a).$$

(Here ϵ is the emissivity of the filament and σ is the Stefan-Boltzman constant.)

So the heat balance equation is

$$\mathbf{C}_f \frac{dT}{dt} = P - \dot{Q}_{\text{transfer}}.$$

Using above results:

$$\mathbf{C}_f \frac{dT}{dt} = \frac{R_{f1}(T)}{\left[R_{f1}(T) + R_{f2}\right]^2} v_o^2 - \frac{1}{\mathfrak{R}}(T - T_a).$$

Inserting the temperature dependence of the resistance:

$$\mathbf{C}_f \frac{dT}{dt} = \frac{R_{f10}[1 + \alpha(T - T_0)]}{\left\{R_{f10}[1 + \alpha(T - T_0)] + R_{f2}\right\}^2} v_o^2 - \frac{1}{\mathfrak{R}}(T - T_a).$$

Re-writing:

$$\mathbf{C}_f \frac{dT}{dt} = \frac{1 + \alpha(T - T_0)}{\left[1 + \frac{R_{f2}}{R_{f10}} + \alpha(T - T_0)\right]^2} \frac{v_o^2}{R_{f10}} - \frac{1}{\mathfrak{R}}(T - T_a).$$

If we assume that the oscillation period is much shorter than the time to reach thermal equilibrium, we suppose that the system reaches a steady state that everages over the heating per cycle by the signal at a given amplitude. Then we say that

$$\mathbf{C}_f \frac{dT}{dt} \approx 0.$$

This requires that

$$\frac{1+\alpha(T-T_0)}{\left[1+\frac{R_{f2}}{R_{f10}}+\alpha(T-T_0)\right]^2}\frac{v_o^2}{R_{f10}}-\frac{1}{\mathfrak{R}}(T-T_a)=0.$$

As we will see, the temperature difference $T-T_0\sim v_o^2$, so the equation above can be simplified to lowest order in v_o^2:

$$\frac{1}{\left[1+\frac{R_{f2}}{R_{f10}}\right]^2}\frac{v_o^2}{R_{f10}}-\frac{1}{\mathfrak{R}}(T-T_a)=0.$$

We'll also assume that the ambient temperature and reference temperature for the filament are the same: $T_a=T_0$.

Then we have:

$$(T-T_0)\approx\frac{1}{\left[1+\frac{R_{f2}}{R_{f10}}\right]^2}\frac{\mathfrak{R}}{R_{f10}}v_o^2\approx\frac{1}{9}\frac{\mathfrak{R}}{R_{f10}}v_o^2,$$

where we have used the critical value of the resistance ratio

$$\frac{R_{f2c}}{R_{f10}}=2.$$

The dynamical equation including temperature dependence of the resistance R_{f1} is:

$$\frac{d^2v_o}{dt^2}-\left(\frac{R_{f2}}{R_{f10}[1+\alpha(T-T_0)]}-2\right)\omega_0\frac{dv_o}{dt}+\omega_0^2v_o=0.$$

Assuming, at least near the onset of oscillations, that the self-heating is modest and thus the correction is small compared to 1:

$$\frac{d^2v_o}{dt^2}-\left(\frac{R_{f2}}{R_{f10}}[1-\alpha(T-T_0)]-2\right)\omega_0\frac{dv_o}{dt}+\omega_0^2v_o\approx0.$$

Substituting for $T-T_0$:

$$\frac{d^2v_o}{dt^2}-\left(\frac{R_{f2}}{R_{f10}}\left[1-\alpha\frac{1}{9}\frac{\mathfrak{R}}{R_{f10}}v_o^2\right]-2\right)\omega_0\frac{dv_o}{dt}+\omega_0^2v_o=0.$$

$$\frac{d^2v_o}{dt^2}-\left(\frac{R_{f2}}{R_{f10}}-2-\frac{R_{f2}}{R_{f10}}\frac{1}{9}\frac{\alpha\mathfrak{R}}{R_{f10}}v_o^2\right)\omega_0\frac{dv_o}{dt}+\omega_0^2v_o=0.$$

$$\frac{d^2v_o}{dt^2}-\left(\frac{R_{f2}}{R_{f10}}-2-\frac{2}{9}\frac{\alpha\mathfrak{R}}{R_{f10}}v_o^2\right)\omega_0\frac{dv_o}{dt}+\omega_0^2v_o=0.$$

where we used in the last term

$$\frac{R_{f2}}{R_{f10}} \approx 2$$

near the onset of oscillations.

Scaled equation

The model equation is

$$\frac{d^2 v_o}{dt^2} - \left(\frac{R_{f2}}{R_{f10}} - 2 - \frac{2}{9}\frac{\alpha\Re}{R_{f10}} v_o^2\right)\omega_0 \frac{dv_o}{dt} + \omega_0^2 v_o = 0.$$

Let's define

$$\epsilon \equiv \frac{R_{f2}}{R_{f10}} - 2$$

$$\delta \equiv \frac{2}{9}\frac{\alpha\Re}{R_{f10}}$$

.

The equation is then written as:

$$\frac{d^2 v_o}{dt^2} - (\epsilon - \delta v_o^2)\omega_0 \frac{dv_o}{dt} + \omega_0^2 v_o = 0.$$

Rescale time $\tilde{t} \equiv \omega_0 t$ and rescale voltage so that $\tilde{v} = \sqrt{\delta} v_0$. Then the equation becomes:

$$\frac{d^2 \tilde{v}}{d\tilde{t}^2} - (\epsilon - \tilde{v}^2)\frac{d\tilde{v}}{d\tilde{t}} + \tilde{v} = 0.$$

This allows a lot of different experimental conidtions to be captured into a single computation of $\tilde{v}(\tilde{t})$. The actual solution is found as:

$$v_o(t) = \frac{1}{\delta^2}\tilde{v}(\omega_o t).$$

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Integrating the dynamical system

The second-order differential equation is equivalent to the following pair of first-order differential equation. (We'll drop the tilde notation for now, assuming that these are the scaled equations.)

$$\frac{dv}{dt} = w.$$

$$\frac{dw}{dt} = (\epsilon - v^2)w - v.$$

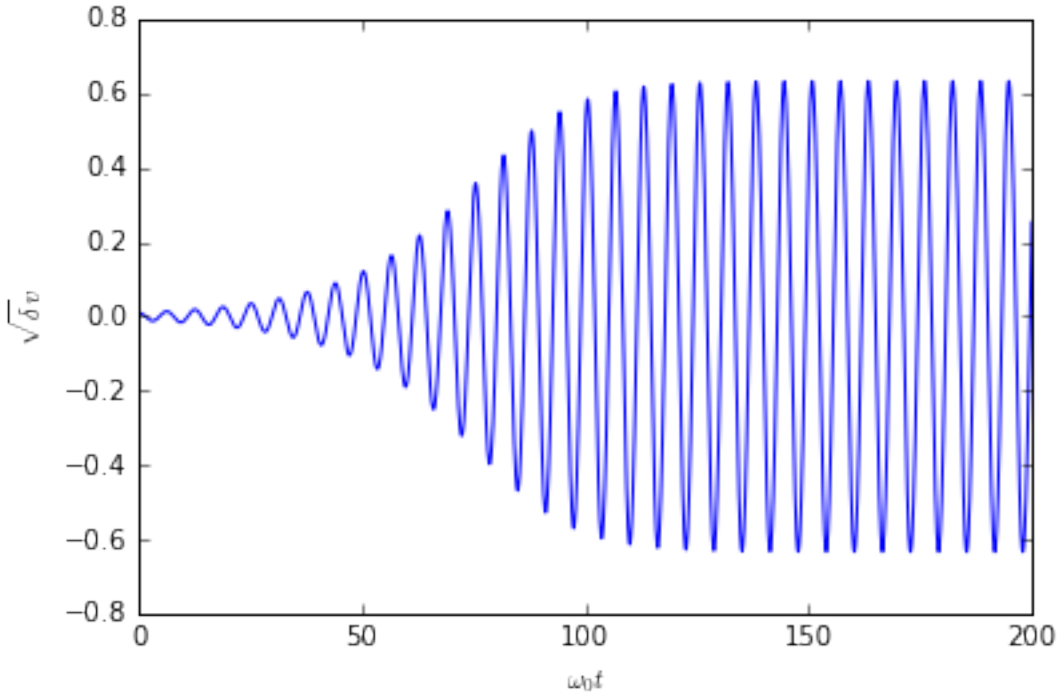
In [41]:

```
% matplotlib inline
# Code to numerically integrate the actual dynamical system
import numpy as np
import matplotlib.pyplot as plt
from scipy import integrate
import math
#
# define the dynamical system to be used by the numerical integrator
def dynsys(xx,tt,epsilon,gamma):
    ff1 = xx[1]
    ff2 = (epsilon-xx[0]**2)*xx[1]-xx[0]
#    print(ff1,ff2)
    return [ff1,ff2]
#
R = 6800. # ohms
C = 0.01E-6 # farads
print('R= ',R,' ohms   C=',C,' farads')
omega0 = 1/(R*C)
f =omega0/(2*math.pi)
print('Frequency {0:6.1f} Hz {1:6.1f} radians/s'.format(f,omega0)) # (obtained 1591
#
Rf10 = 54. # ohms - Radio Shack 12V 25mA mini lamp part number 272-1141
alpha = 0.0045 # K^(-1) tungsten - http://hyperphysics.phy-astr.gsu.edu/hbase/Table
rho = 5.6E-8 # ohm-m tungsten - same reference as above
T0 = 20. # celsius
Ta = 20. # celsius
# filament characteristics
Rfil = (2500-293)/(12*0.025) # K/W assume 12volt 25mA lamp reaches 2500K
print('Thermal resistance {0:6.1f} K/W'.format(Rfil)) # obtained value 7357 K/W
L = 2.0E-2 # filament length in m (a guess after using magnifier to inspect filamen
A = rho*L/Rf10 # filament cross-sectional area
V = L*A # filament volume
rhomass = 19.25E3 # Kg/m^3 density of tungsten
m = rhomass*V*1000 # g mass of filament
print('Filament mass {0:6.3e} grams'.format(m)) # obtained 7.99E-6 g
specificheat = 0.134 # J/gK tungsten
Cfil = specificheat*m # J/K heat capacity of filament
print('Heat capacity {0:6.3e} J/K'.format(Cfil)) # obtained value 1.07E-6 J/K
tauthermal = Cfil*Rfil # s filament time constant
print('Thermal time constant {0:6.3e} seconds'.format(tauthermal)) # obtained value
#
delta = alpha*Rfil/Rf10 # volts^(-2)
print('Nonlinear parameter delta = {0:6.4f} volt^-2'.format(delta)) # obtained 0.61
print('Voltage scale = {0:6.3f} volt'.format(1/math.sqrt(delta))) # obtained 0.613
#
#Rf2 = 113.4 # ohms
#Rf2 = 108.6 # ohms
#epsilon = Rf2/Rf10-2.
```

```
#
# Initial conditions
v_0 = 0.01
w_0 = 0.0
#
epsilon = 0.1 # specify epsilon directly
print('epsilon = ',epsilon)
timestep = 0.01
timerange = 200
Nstep = int(timerange/timestep)
a_tt = np.arange(0,timerange,timestep) # time array
# See https://docs.scipy.org/doc/scipy-0.18.1/reference/generated/scipy.integrate.oc
# Also https://nathantypanski.com/blog/2014-08-23-ode-solver-py.html
gamma = 0.
a_sol = integrate.odeint(dynsys,[v_0,w_0],a_tt,args=(epsilon,gamma))
vv = a_sol[0:Nstep,0]
plt.plot(a_tt,vv)
plt.xlabel('$\omega_0 t$')
plt.ylabel('$\sqrt{\delta}v$')
#plt.annotate('eps = {0:4.2f}'.format(epsilon),xy=(5,0.75))
plt.show()
#
# Now compute and plot epsilon dependence of final amplitude
# along with theoretical value 2*sqrt(epsilon)
timestep = 0.01
timerange = 500 # need a longer time range for smaller epsilon
Nstep = int(timerange/timestep)
a_tt = np.arange(0,timerange,timestep) # time array
a_eps = []
a_vmax = []
a_thval = []
epsilon = 0.
for i in range(0,50):
  epsilon = epsilon + 0.01
  a_eps = np.append(a_eps,epsilon)
  a_sol = integrate.odeint(dynsys,[v_0,w_0],a_tt,args=(epsilon,gamma))
  vv = a_sol[0:Nstep,0]
  vmax = np.amax(vv)
  a_vmax = np.append(a_vmax,vmax)
  thval = 2*math.sqrt(epsilon)
  a_thval = np.append(a_thval,thval)
  print('epsilon={0:0.4f},vmax={1:3.4f},thval={2:3.4f}'.format(epsilon,vmax,thval))
plt.plot(a_eps,a_vmax,'ro',a_eps,a_thval,'k')
plt.xlabel('epsilon')
plt.ylabel('vmax & theor. value')
plt.show()
#print(vv)
```

```
R=  6800.0  ohms   C= 1e-08  farads
Frequency 2340.5 Hz 14705.9 radians/s
Thermal resistance 7356.7 K/W
Filament mass 7.985e-06 grams
Heat capacity 1.070e-06 J/K
```

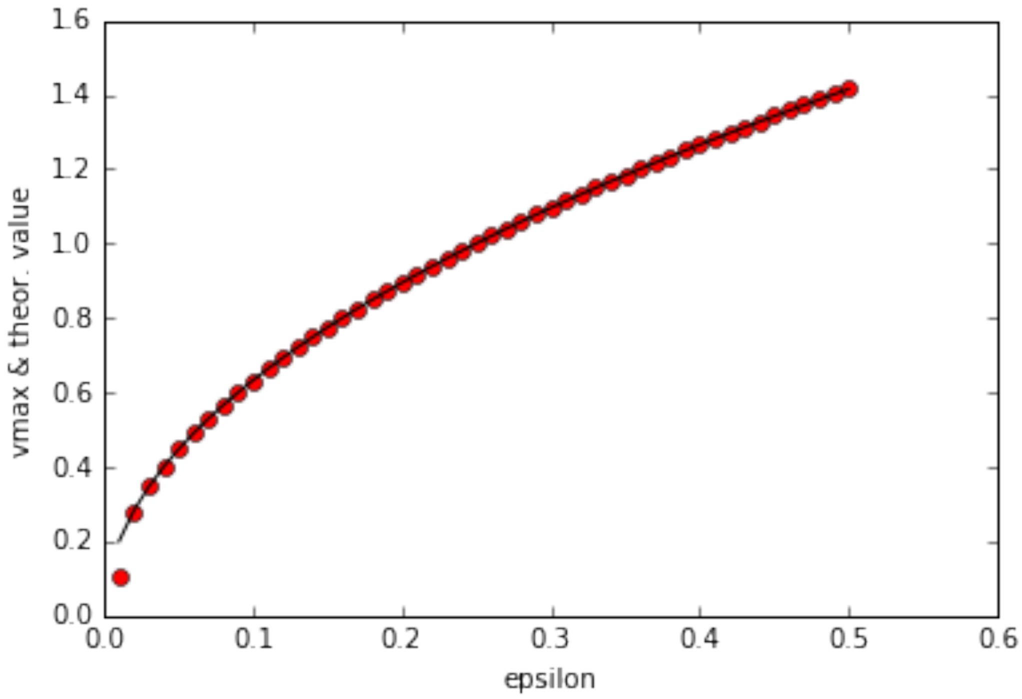
```
Thermal time constant 7.872e-03 seconds
Nonlinear parameter delta = 0.6131 volt^-2
Voltage scale =  1.277 volt
epsilon =  0.1
```




```
epsilon=0.0100,vmax=0.1027,thval=0.2000
epsilon=0.0200,vmax=0.2775,thval=0.2828
epsilon=0.0300,vmax=0.3463,thval=0.3464
epsilon=0.0400,vmax=0.4000,thval=0.4000
epsilon=0.0500,vmax=0.4472,thval=0.4472
epsilon=0.0600,vmax=0.4899,thval=0.4899
epsilon=0.0700,vmax=0.5292,thval=0.5292
epsilon=0.0800,vmax=0.5657,thval=0.5657
epsilon=0.0900,vmax=0.6000,thval=0.6000
epsilon=0.1000,vmax=0.6325,thval=0.6325
epsilon=0.1100,vmax=0.6634,thval=0.6633
epsilon=0.1200,vmax=0.6929,thval=0.6928
epsilon=0.1300,vmax=0.7212,thval=0.7211
epsilon=0.1400,vmax=0.7484,thval=0.7483
epsilon=0.1500,vmax=0.7747,thval=0.7746
epsilon=0.1600,vmax=0.8001,thval=0.8000
epsilon=0.1700,vmax=0.8247,thval=0.8246
epsilon=0.1800,vmax=0.8487,thval=0.8485
epsilon=0.1900,vmax=0.8719,thval=0.8718
epsilon=0.2000,vmax=0.8946,thval=0.8944
epsilon=0.2100,vmax=0.9167,thval=0.9165
epsilon=0.2200,vmax=0.9383,thval=0.9381
epsilon=0.2300,vmax=0.9594,thval=0.9592
epsilon=0.2400,vmax=0.9801,thval=0.9798
epsilon=0.2500,vmax=1.0003,thval=1.0000
epsilon=0.2600,vmax=1.0202,thval=1.0198
epsilon=0.2700,vmax=1.0396,thval=1.0392
epsilon=0.2800,vmax=1.0587,thval=1.0583
epsilon=0.2900,vmax=1.0775,thval=1.0770
epsilon=0.3000,vmax=1.0960,thval=1.0954
epsilon=0.3100,vmax=1.1141,thval=1.1136
epsilon=0.3200,vmax=1.1320,thval=1.1314
epsilon=0.3300,vmax=1.1496,thval=1.1489
```

```
epsilon=0.3300,vmax=1.1496,thval=1.1489
epsilon=0.3400,vmax=1.1669,thval=1.1662
epsilon=0.3500,vmax=1.1840,thval=1.1832
epsilon=0.3600,vmax=1.2008,thval=1.2000
epsilon=0.3700,vmax=1.2174,thval=1.2166
epsilon=0.3800,vmax=1.2338,thval=1.2329
epsilon=0.3900,vmax=1.2500,thval=1.2490
epsilon=0.4000,vmax=1.2659,thval=1.2649
epsilon=0.4100,vmax=1.2817,thval=1.2806
epsilon=0.4200,vmax=1.2973,thval=1.2961
epsilon=0.4300,vmax=1.3127,thval=1.3115
epsilon=0.4400,vmax=1.3279,thval=1.3266
epsilon=0.4500,vmax=1.3430,thval=1.3416
epsilon=0.4600,vmax=1.3579,thval=1.3565
epsilon=0.4700,vmax=1.3726,thval=1.3711
epsilon=0.4800,vmax=1.3872,thval=1.3856
epsilon=0.4900,vmax=1.4017,thval=1.4000

epsilon=0.5000,vmax=1.4160,thval=1.4142
```



Alternate rescaling to obtain the van der Pol equation

In order to connect with some of the literature, it is useful to see a different scaling that depends on ϵ. This leads to the famous van der Pol equation. Unfortunately, the bifurcation behvaior is masked because the scaling becomes singular at $\epsilon = 0$.

Starting with the fully dimensioned equation:

$$\frac{d^2 v_o}{dt^2} - \left(\frac{R_{f2}}{R_{f10}} - 2 - \frac{2}{9} \frac{\alpha \mathfrak{R}}{R_{f10}} v_o^2 \right) \omega_0 \frac{dv_o}{dt} + \omega_0^2 v_o = 0.$$

Again define $\epsilon \equiv \frac{R_{f2}}{R_{f10}} - 2$ and $\delta \equiv \frac{2}{9} \frac{\alpha \mathfrak{R}}{R_{f10}}$.

$$\frac{d^2 v_o}{dt^2} - (\epsilon - \delta v_o^2) \omega_0 \frac{dv_o}{dt} + \omega_0^2 v_o = 0.$$

Provided that $\epsilon \neq 0$ we can write:

$$\frac{d^2 v_o}{dt^2} - \epsilon(1 - \frac{\delta}{\epsilon} v_o^2)\omega_0 \frac{dv_o}{dt} + \omega_0^2 v_o = 0.$$

Rescale time $\tilde{t} \equiv \omega_0 t$ and $\tilde{v} = \sqrt{\frac{\delta}{\epsilon}} v_0$. Then the equation becomes:

$$\frac{d^2 \tilde{v}}{d\tilde{t}^2} - \epsilon(1 - \tilde{v}^2)\frac{d\tilde{v}}{d\tilde{t}} + \tilde{v} = 0.$$

This is the standard form of the van der Pol equation. Notice, however, that the scale factor $\sqrt{\frac{\delta}{\epsilon}}$ becomes singular at $\epsilon = 0$ at the same point that the actual output voltage amplitude is just about to rise from a value of zero.

Numerical integration of the full dynamical system

The actual dynamical system with a self-heating feedback resistor consists of equations for the time derivatives of the variables $v \equiv v_o$, $w \equiv \frac{dv}{dt}$, and T. We can return to this more "fundamental" form of the model and explore the effect of the dynamics of heat transfer, reflected by the thermal time constant.

$$\frac{dv}{dt} = w.$$

$$\frac{dw}{dt} = \left(\frac{R_{f2}}{R_{f10}[1 + \alpha(T - T_0)]} - 2\right)\omega_0 w - \omega_0^2 v.$$

$$\frac{dT}{dt} = \frac{1}{\mathbf{C}_f}\frac{1 + \alpha(T - T_0)}{\left[1 + \frac{R_{f2}}{R_{f10}} + \alpha(T - T_0)\right]^2}\frac{v^2}{R_{f10}} - \frac{1}{\mathfrak{R}\mathbf{C}_f}(T - T_a).$$

Note that this admits a steady state solution for all values of the "control parameter" R_{f2}:

$v = 0, w = \frac{dv}{dt} = 0, \text{ and } T = T_a.$

Thus when an oscillation appears, it is a new solution that is said to "bifurcate" from the steady-state solution.

In [ ]:

```
# TO BE ADDED: rescaling and code to integrate the above system of three equations.
```

Method of Averaging

NOTE: THIS PART IS STILL IN PREPARATION AND SHOULD BE IGNORED FOR NOW. EVENTUALLY IT WILL LEAD TO THE PREDICTION THAT THE SCALED AMPLITUTE GROWS LIKE $2\sqrt{\epsilon}$.

We will return to the single nonlinear equation in v_o.

$$\frac{d^2 v_o}{dt^2} - (\epsilon - \delta v_o^2)\omega_0 \frac{dv_o}{dt} + \omega_0^2 v_o = 0.$$

where

$$\epsilon \equiv \frac{R_{f2}}{R_{f10}} - 2.$$

$$\delta \equiv \frac{2}{9}\frac{\alpha \mathfrak{R}}{R_{f10}}.$$

We are going to assume that the the solution will evolve from the simple sinsoidal solution when the nonlinear term is turned on $\delta > 0$.

$$v_o(t) = A(t)\cos[\omega_0 t + \phi(t)].$$

$$\frac{dv_o(t)}{dt} = -\omega_0 A(t)\sin[\omega_0 t + \phi(t)].$$

Our goal is to find equations for the amplitude function $A(t)$ and the phase function $\phi(t)$.

To be continued...